

# ALMOST ALL LAGRANGIAN TORUS ORBITS IN $\mathbb{C}P^n$ ARE NOT HAMILTONIAN VOLUME MINIMIZING

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**ABSTRACT.** All principal orbits of the standard Hamiltonian  $T^n$ -action on the complex projective space  $\mathbb{C}P^n$  are Lagrangian tori. In this article, we prove that most of them are not volume minimizing under Hamiltonian isotopies of  $\mathbb{C}P^n$  if the complex dimension  $n$  is greater than two, although they are Hamiltonian minimal and Hamiltonian stable.

## 1. INTRODUCTION

The classical isoperimetric inequality for a simple closed curve  $L$  in  $\mathbb{R}^2$  (resp. the unit two-sphere  $S^2$ ) states that

$$l(L)^2 \geq 4\pi A \quad (\text{resp. } l(L)^2 \geq 4\pi A - A^2),$$

where  $l(L)$  is the length of  $L$  and  $A$  the area of the disc enclosed by  $L$ . Moreover, the equality holds if and only if  $L$  is a round circle. In other words, a round circle  $L = S^1$  in  $\mathbb{R}^2$  (or  $S^2$ ) has least length when we deform  $L$  in such a way that the enclosed area  $A$  is unchanged. Notice that without this last constraint we can easily reduce the length of  $S^1$  by deforming it to the normal direction.

In papers [8] and [9], Y.-G. Oh proposed a higher dimensional analogue of such a phenomenon from the symplectic geometrical viewpoint and introduced several concepts. Let us review the settings. Let  $(M, \omega, J)$  be a Kähler manifold. A submanifold  $L$  of  $M$  is said to be *Lagrangian* if  $\omega|_{TL} \equiv 0$  and  $\dim_{\mathbb{R}} L = \dim_{\mathbb{C}} M$ . This condition is equivalent to the existence of an orthogonal decomposition

$$T_p M = T_p L \oplus J(T_p L)$$

for any  $p \in L$ . Throughout this article all Lagrangian submanifolds are assumed to be connected, embedded, closed and equipped with the

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induced Riemannian metric from the ambient manifold  $M$ . We denote by  $\text{Vol}(L)$  the volume of  $L$  with respect to the metric.

Notice that  $\mathbb{R}^2 \cong \mathbb{C}$  and  $S^2$  are one-dimensional Kähler manifolds and a simple closed curve in them is a Lagrangian submanifold. The constraint that  $A$  is constant is generalised to the deformation of a Lagrangian submanifold  $L$  under *Hamiltonian isotopies* explained below. By definition, we have the linear isomorphism defined by

$$\Gamma(T^\perp L) \ni V \longmapsto \alpha_V := \omega(V, \cdot)|_{TL} \in \Omega^1(L),$$

where  $T^\perp L (\cong J(TL))$  denotes the normal bundle of  $L \subset M$  and  $\Omega^1(L)$  the set of all one-forms on  $L$ . A variational vector field  $V \in \Gamma(T^\perp L)$  of  $L$  is called a *Hamiltonian variation* if  $\alpha_V$  is exact. It implies that the infinitesimal deformation of  $L$  with the vector field  $V$  preserves the Lagrangian constraint. The following definitions are due to Oh.

**Definition 1** ([9], [8]). Let  $L$  be a Lagrangian submanifold of a Kähler manifold  $(M, \omega, J)$ .

- (1)  $L \subset M$  is said to be *Hamiltonian minimal* if it satisfies that

$$\left. \frac{d}{dt} \text{Vol}(L_t) \right|_{t=0} = 0$$

for any smooth deformation  $\{L_t\}_{-\epsilon < t < \epsilon}$  of  $L = L_0$  with a Hamiltonian variation  $V = \left. \frac{dL_t}{dt} \right|_{t=0}$ .

- (2) Suppose that  $L \subset M$  is Hamiltonian minimal. Then  $L$  is said to be *Hamiltonian stable* if it satisfies that

$$\left. \frac{d^2}{dt^2} \text{Vol}(L_t) \right|_{t=0} \geq 0$$

for any smooth deformation  $\{L_t\}_{-\epsilon < t < \epsilon}$  of  $L = L_0$  with a Hamiltonian variation  $V = \left. \frac{dL_t}{dt} \right|_{t=0}$ .

- (3)  $L \subset M$  is said to be *Hamiltonian volume minimizing* if

$$\text{Vol}(\phi(L)) \geq \text{Vol}(L)$$

holds for any  $\phi \in \text{Ham}_c(M, \omega)$ , which is the set of all compactly supported Hamiltonian diffeomorphisms of  $(M, \omega)$ .

A diffeomorphism  $\phi$  of  $(M, \omega)$  is called *Hamiltonian* if  $\phi$  is the time-one map of the flow  $\{\phi_H^t\}_{0 \leq t \leq 1}$ ,  $\phi_H^0 = \text{id}_M$ , of the (time-dependent) Hamiltonian vector field  $X_{H_t}$  defined by a compactly supported Hamiltonian function  $H \in C_c^\infty([0, 1] \times M)$ . The isotopy  $\{\phi_H^t\}_{0 \leq t \leq 1}$  is called a *Hamiltonian isotopy* of  $M$ . It is easy to see that  $(\phi_H^t)^* \omega = \omega$ . Note that a (time-independent) Hamiltonian vector field on  $M$  gives rise to a Hamiltonian variation of a Lagrangian submanifold  $L \subset M$ .

At present we know only a few non-trivial examples of Hamiltonian volume minimizing Lagrangian submanifolds except for special Lagrangian submanifolds; the real form  $\mathbb{R}P^n \subset \mathbb{C}P^n$ , [8], the product of the great circles in  $S^2 \times S^2$ , [6], and the totally geodesic Lagrangian sphere  $S^{2n-1}$  in the complex hyperquadric  $Q_{2n-1}(\mathbb{C})$ , [7].

The most fundamental example of symplectic manifolds is the linear complex space  $\mathbb{C}^n$  equipped with the standard symplectic structure  $\omega_0 := dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ . Its standard complex structure and  $\omega_0$  define the standard Euclidean metric on  $\mathbb{C}^n \cong \mathbb{R}^{2n}$ . We denote by  $S^1(b) \subset \mathbb{R}^2 \cong \mathbb{C}$  the boundary of a round disc with area  $b$  centred at the origin, i.e., the radius of  $S^1(b)$  is  $\sqrt{b/\pi}$ . For positive real numbers  $b_1, \dots, b_n > 0$ , the *product torus* (or *elementary torus*, see [3])

$$T(\mathbf{b}) = T(b_1, \dots, b_n) := S^1(b_1) \times \cdots \times S^1(b_n) \subset \mathbb{C}^n$$

is a typical example of Lagrangian submanifolds of  $\mathbb{C}^n$ . Here we denote  $N(\mathbf{b}) := \#\{b_1, \dots, b_n\}$ , e.g.,  $N(\mathbf{b}) = 3$  for  $\mathbf{b} = (1, 2, 2, 4)$ .

We can easily check, using the first variation formula (see [9, p. 178]), that  $L \subset M$  is Hamiltonian minimal if and only if the equation  $\delta\alpha_H = 0$  holds on  $L$ , where  $\delta$  and  $H$  are the codifferential operator on  $L$  and the mean curvature vector of  $L$ , respectively. Hence,  $T(\mathbf{b}) \subset \mathbb{C}^n$  is Hamiltonian minimal. Using his second variation formula [9, Theorem 3.4], Oh proved that the torus  $T(\mathbf{b}) \subset \mathbb{C}^n$  is a Hamiltonian stable Lagrangian submanifold (see [9, Theorem 4.1]). Moreover, the isoperimetric inequality for closed curves in  $\mathbb{R}^2$  states that  $T(b_1) \subset \mathbb{C}$  is Hamiltonian volume minimizing. Based on these results, Oh proposed the following

**Conjecture 2** (Oh [9], p.192). The Lagrangian torus  $T(\mathbf{b})$  in  $\mathbb{C}^n$  is Hamiltonian volume minimizing.

In a sense, Conjecture 2 is regarded as a symplectic higher dimensional analogue of the isoperimetric inequality in  $\mathbb{R}^2$ . Though the statement is quite natural, it turned out to be false for  $n \geq 3$ . Indeed, C. Viterbo [12, p.419] has already pointed out that  $T(1, 2, 2)$  and  $T(1, 2, 3)$  are Hamiltonian isotopic based on a remarkable result by Chekanov [3, Theorem A], see Section 2. Namely, the second one is not Hamiltonian volume minimizing. Furthermore, in Section 2 we prove

**Corollary 3.** *Let  $\mathbf{b} \in (\mathbb{R}_{>0})^n$ . If  $N(\mathbf{b}) \geq 3$ , then the Lagrangian torus  $T(\mathbf{b}) \subset \mathbb{C}^n$  is not Hamiltonian volume minimizing.*

If  $n \geq 3$ , then the set

$$\{\mathbf{b} \in (\mathbb{R}_{>0})^n \mid N(\mathbf{b}) \geq 3\}$$

is an open dense subset of  $(\mathbb{R}_{>0})^n$ , and hence almost all product tori in  $\mathbb{C}^n$  ( $n \geq 3$ ) are *not* Hamiltonian volume minimizing. Notice that  $T(\mathbf{b})$  is represented as  $\mu_0^{-1}(b_1/2\pi, \dots, b_n/2\pi)$ , where

$$\mu_0(x_1, \dots, x_n, y_1, \dots, y_n) = \left( \frac{1}{2}(x_1^2 + y_1^2), \dots, \frac{1}{2}(x_n^2 + y_n^2) \right)$$

is the moment map  $\mu_0 : \mathbb{C}^n \rightarrow (\mathbb{R}_{\geq 0})^n$  associated with the standard Hamiltonian action by the real torus  $T^n \subset (\mathbb{C}^\times)^n$  on  $\mathbb{C}^n$ .

Similarly, the complex projective space  $(\mathbb{C}P^n, J_{\text{std}})$  equipped with the standard Fubini-Study Kähler form  $\omega_{\text{FS}}$  admits an effective Hamiltonian  $T^n$ -action. Each principal orbit is a flat Lagrangian torus in  $\mathbb{C}P^n$  like product one in  $\mathbb{C}^n$ . As for its Hamiltonian minimality and Hamiltonian stability, the second author previously proved

**Proposition 4** ([10], Section 4). *Any Lagrangian torus orbit  $T^n$  in  $(\mathbb{C}P^n, \omega_{\text{FS}}, J_{\text{std}})$  is Hamiltonian minimal and Hamiltonian stable.*

Hence, it is worthwhile to determine whether each Lagrangian torus orbit  $T^n$  is Hamiltonian volume minimizing or not. The following is the main result of the present article, which provides a negative solution for the problem (see Conjecture 1.4 in [10]).

**Theorem 5.** *If  $n \geq 3$ , then almost all Lagrangian torus orbits in  $\mathbb{C}P^n$  are not Hamiltonian volume minimizing.*

The proof, which is given in Section 3, is based on a recent result of Chekanov and Schlenk [4] which gives a refinement of the Chekanov's one mentioned above.

In general, Darboux's theorem says that any point in a symplectic manifold  $(M, \omega)$  possesses a neighbourhood which is isomorphic to a neighbourhood of the origin of  $(\mathbb{C}^n, \omega_0)$ . Then the Chekanov-Schlenk's theorem ensures any symplectic manifold the existence of a pair of Lagrangian tori which are mutually Hamiltonian isotopic and not intersect. Furthermore, in the class of compact toric symplectic manifolds, we can regard the Chekanov-Schlenk's theorem as a local model of a  $T^n$ -fixed point of such a manifold. Although the result is weaker than the case of  $\mathbb{C}P^n$ , this observation yields the following

**Theorem 6.** *Let  $(M, \omega, J)$  be a complex  $n$ -dimensional compact toric Kähler manifold. If  $n \geq 3$ , then there exists a toric fibre of  $M$  (indeed, infinitely many) which is not Hamiltonian volume minimizing.*

We prove it in Section 4. Notice that toric fibres in Theorem 6 are all Hamiltonian minimal (see Section 4).

2. PRODUCT TORI IN  $\mathbb{C}^n$  AND CHEKANOV-SCHLENK'S THEOREM

In this section, we shall consider the case of  $(\mathbb{C}^n, \omega_0)$ . For  $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{R}_{>0})^n$ , we use the following notations:

$$\underline{\mathbf{a}} = \min\{a_i \mid 1 \leq i \leq n\}, \quad \overline{\mathbf{a}} = \max\{a_i \mid 1 \leq i \leq n\}, \quad |\mathbf{a}| = \sum_{i=1}^n a_i,$$

$$m(\mathbf{a}) = \#\{i \mid a_i = \underline{\mathbf{a}}\}, \quad \|\mathbf{a}\| = |\mathbf{a}| + \underline{\mathbf{a}}, \quad \|\|\mathbf{a}\|\| = |\mathbf{a}| + \overline{\mathbf{a}},$$

$$\Gamma(\mathbf{a}) = \text{span}_{\mathbb{Z}}\langle a_1 - \underline{\mathbf{a}}, \dots, a_n - \underline{\mathbf{a}} \rangle \subset \mathbb{R}.$$

For  $\mathbf{a}, \mathbf{a}' \in (\mathbb{R}_{>0})^n$ , we denote  $\mathbf{a} \simeq \mathbf{a}'$  if

$$(\underline{\mathbf{a}}, m(\mathbf{a}), \Gamma(\mathbf{a})) = (\underline{\mathbf{a}'}, m(\mathbf{a}'), \Gamma(\mathbf{a}')),$$

and consider the set

$$\tilde{\Delta}_s := \left\{ (a_1, \dots, a_n) \in (\mathbb{R}_{\geq 0})^n \mid \sum_{i=1}^n a_i < s \right\}.$$

Notice that  $\mu_0^{-1}(\tilde{\Delta}_s)$  is the open ball in  $\mathbb{C}^n$  with radius  $\sqrt{2s}$  centred at the origin. Let  $L$  and  $L'$  be Lagrangian submanifolds of  $(M, \omega)$ . Then  $L$  is said to be *Hamiltonian isotopic* to  $L'$  if there exists  $\phi \in \text{Ham}_c(M, \omega)$  such that  $\phi(L) = L'$ . The following result is fundamental for the arguments of this article.

**Theorem 7** (Chekanov [3]). *Let  $\mathbf{a}, \mathbf{a}' \in (\mathbb{R}_{>0})^n$ . A product torus  $T(\mathbf{a})$  of  $(\mathbb{C}^n, \omega_0)$  is Hamiltonian isotopic to  $T(\mathbf{a}')$  if and only if  $\mathbf{a} \simeq \mathbf{a}'$  holds.*

**Proposition 8** (Corollary 3). *If  $N(\mathbf{a}) \geq 3$ , then the product torus  $\mu_0^{-1}(\mathbf{a}) = T(2\pi\mathbf{a}) \subset \mathbb{C}^n$  is not Hamiltonian volume minimizing.*

*Proof.* For  $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{R}_{>0})^n$ , by assumption, there exist numbers  $i, j \in \{1, 2, \dots, n\}$  such that  $\underline{\mathbf{a}} < a_i < a_j$ . We define a new  $\mathbf{a}'$  as

$$\mathbf{a}' = (a'_1, \dots, a'_n) := (a_1, \dots, a_{j-1}, a_j - a_i + \underline{\mathbf{a}}, a_{j+1}, \dots, a_n).$$

Then we have  $\mathbf{a} \simeq \mathbf{a}'$  and  $\|\mathbf{a}\| > \|\mathbf{a}'\|$ . Since  $\Pi_i a_i > \Pi_i a'_i$ , Theorem 7 implies that  $\mu_0^{-1}(\mathbf{a})$  is not Hamiltonian volume minimizing.  $\square$

Furthermore, the size of the support of a Hamiltonian isotopy connecting two product tori in Theorem 7 has precisely estimated as follows. This estimation is essential to treat the case of  $\mathbb{C}P^n$ .

**Theorem 9** (Chekanov-Schlenk [4], Theorem 1.1). *For  $\mathbf{a}, \mathbf{a}' \in (\mathbb{R}_{>0})^n$ , suppose that  $\mathbf{a} \simeq \mathbf{a}'$ . Let  $s$  be a positive number satisfying that  $s > \max\{\|\mathbf{a}\|, \|\mathbf{a}'\|\}$ . Then there exists a smooth Hamiltonian function  $H : [0, 1] \times \mathbb{C}^n \rightarrow \mathbb{R}$  satisfying the following:*

- (1)  $\text{Supp}(H) \subset [0, 1] \times \mu_0^{-1}(\tilde{\Delta}_s)$ .
- (2)  $\phi_H^1(\mu_0^{-1}(\mathbf{a})) = \mu_0^{-1}(\mathbf{a}')$ .

### 3. LAGRANGIAN TORUS ORBITS IN $\mathbb{C}P^n$

In this section, we shall treat the case of  $\mathbb{C}P^n$  and prove the main theorem.

**3.1.  $\mathbf{e}_i$ -action-angle coordinates.** Let us consider  $\mathbb{R}^n$  and take an orthonormal basis

$$\mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \mathbf{e}_n := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

of  $\mathbb{R}^n$  and set  $\Delta := \{(a_1, \dots, a_n) \in (\mathbb{R}_{\geq 0})^n \mid \sum_{i=1}^n a_i \leq 1\}$ . For a notational reason, we put  $\mathbf{e}_0 := {}^t(0, 0, \dots, 0) \in \mathbb{R}^n$ . The symplectic toric manifold corresponding to the polytope  $\Delta$  is nothing but the  $n$ -dimensional complex projective space  $(\mathbb{C}P^n, \omega_{\text{FS}}, \mu)$ .

We first examine the coordinate neighbourhood given by

$$U_0 = \{[z_0 : z_1 : \dots : z_n] \mid z_0 \neq 0\} \xrightarrow{\sim} \mathbb{C}^n, \quad [z_0 : \dots : z_n] \mapsto \left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right).$$

We put

$$r_0^i := \left| \frac{z_i}{z_0} \right|, \quad \theta_0^i := \arg \frac{z_i}{z_0}$$

for  $i = 1, \dots, n$ . Then the moment map associated with the standard Hamiltonian  $T^n$ -action on  $(\mathbb{C}P^n, \omega_{\text{FS}}, \mu)$  is represented as

$$\mu = \sum_{i=1}^n u_0^i \mathbf{e}_i, \quad u_0^i := \frac{(r_0^i)^2}{1 + \sum_{j=1}^n (r_0^j)^2}.$$

Here we introduce the coordinates defined by

$$x_0^i := \sqrt{2u_0^i} \cos \theta_0^i, \quad y_0^i := \sqrt{2u_0^i} \sin \theta_0^i.$$

Then, on  $U_0$  the symplectic structure  $\omega_{\text{FS}}$  and the moment map  $\mu$  are expressed as follows:

$$\omega_{\text{FS}}|_{U_0} = \sum_{i=1}^n du_0^i \wedge d\theta_0^i = \sum_{i=1}^n dx_0^i \wedge dy_0^i, \quad \mu|_{U_0} = \frac{1}{2} \sum_{i=1}^n \{(x_0^i)^2 + (y_0^i)^2\} \mathbf{e}_i.$$

Hence we have an isomorphism

$$(U_0, \omega_{\text{FS}}|_{U_0}, \mu|_{U_0}) \cong (\mu_0^{-1}(\tilde{\Delta}_1), \omega_0, \mu_0)$$

as Hamiltonian  $T^n$ -spaces. We call the coordinates  $(u_0^1, \dots, u_0^n, \theta_0^1, \dots, \theta_0^n)$   $\mathbf{e}_0$ -action-angle coordinates.

Similarly, we examine the coordinate neighbourhood given by

$$U_1 = \{[z_0 : z_1 : \dots : z_n] \mid z_1 \neq 0\} \xrightarrow{\sim} \mathbb{C}^n, \quad [z_0 : \dots : z_n] \mapsto \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}, \dots, \frac{z_n}{z_1}\right).$$

(The case where  $i \geq 2$  is similar.) We put

$$r_1^1 := \left| \frac{z_0}{z_1} \right|, \quad r_1^i := \left| \frac{z_i}{z_1} \right| \quad (i \geq 2), \quad \theta_1^1 := \arg \frac{z_0}{z_1}, \quad \theta_1^i := \arg \frac{z_i}{z_1} \quad (i \geq 2),$$

Then we have

$$\mu = u_1^1(\mathbf{e}_0 - \mathbf{e}_1) + \sum_{i=2}^n u_1^i(\mathbf{e}_i - \mathbf{e}_1) + \mathbf{e}_1, \quad u_1^i := \frac{(r_1^i)^2}{1 + \sum_{j=1}^n (r_1^j)^2} \quad (i \geq 1).$$

We also introduce the coordinates defined by

$$x_1^i := \sqrt{2u_1^i} \cos \theta_1^i, \quad y_1^i := \sqrt{2u_1^i} \sin \theta_1^i.$$

Then, on  $U_1$  we have

$$\omega_{\text{FS}}|_{U_1} = \sum_{i=1}^n du_1^i \wedge d\theta_1^i = \sum_{i=1}^n dx_1^i \wedge dy_1^i,$$

$$\mu|_{U_1} = \frac{1}{2} \{ (x_1^1)^2 + (y_1^1)^2 \} (\mathbf{e}_0 - \mathbf{e}_1) + \frac{1}{2} \sum_{i=2}^n \{ (x_1^i)^2 + (y_1^i)^2 \} (\mathbf{e}_i - \mathbf{e}_1) + \mathbf{e}_1.$$

Hence we obtain an isomorphism

$$(U_1, \omega_{\text{FS}}|_{U_1}, \mu|_{U_1}) \cong (\mu_0^{-1}(\tilde{\Delta}_1), \omega_0, \mu_0)$$

as Hamiltonian  $T^n$ -spaces. Moreover, on  $U_0 \cap U_1$ ,

$$r_0^1 = \frac{1}{r_1^1}, \quad r_0^j = \frac{r_1^j}{r_1^1} \quad (j \geq 2), \quad \theta_0^1 = -\theta_1^1, \quad \theta_0^j = \theta_1^j - \theta_1^1 \quad (j \geq 2)$$

hold. Then we have

$$u_0^1 = \frac{1}{1 + \sum_{j=1}^n (r_1^j)^2} = 1 - \sum_{j=1}^n u_1^j, \quad u_0^j = u_1^j \quad (j \geq 2)$$

and we can easily check that

$$\mu = \sum_{j=1}^n u_0^j \mathbf{e}_j = u_1^1(\mathbf{e}_0 - \mathbf{e}_1) + \sum_{i=2}^n u_1^i(\mathbf{e}_i - \mathbf{e}_1) + \mathbf{e}_1.$$

Similarly, the  $\mathbf{e}_i$ -action coordinates  $(u_i^1, \dots, u_i^n)$  satisfies that

$$u_i^j = u_0^j \quad (i \neq j), \quad u_i^i = 1 - \sum_{j=1}^n u_0^j$$

on  $U_0 \cap U_i$  and

$$\mu|_{U_0 \cap U_i} = \sum_{j=1}^i u_i^j (\mathbf{e}_{j-1} - \mathbf{e}_i) + \sum_{j=i+1}^n u_i^j (\mathbf{e}_j - \mathbf{e}_i) + \mathbf{e}_i.$$

Hence, on  $U_0 \cap U_i$  the symplectic structure  $\omega_{\text{FS}}$  described as

$$\omega_{\text{FS}}|_{U_0 \cap U_i} = \sum_{j=1}^n du_0^j \wedge d\theta_0^j = \sum_{j=1}^n du_i^j \wedge d\theta_i^j.$$

**3.2. Volume of a Lagrangian torus orbit in  $\mathbb{C}P^n$ .** Recall that the moment map  $\mu : \mathbb{C}P^n \rightarrow \Delta$  is associated with the standard Hamiltonian  $T^n$ -action on  $(\mathbb{C}P^n, \omega_{\text{FS}})$ . The volume of a  $T^n$ -orbit  $\mu^{-1}(p)$ ,  $p \in \text{Int}(\Delta)$ , can be calculated by using Abreu's symplectic potential (see Section 4). Let  $(u_0^1, \dots, u_0^n)$  be the  $\mathbf{e}_0$ -action coordinates of  $p$ . Then we obtain

$$(\text{Vol}(\mu^{-1}(p)))^2 = C \left( 1 - \sum_{j=1}^n u_0^j \right) \prod_{k=1}^n u_0^k,$$

where  $C$  is a positive constant. As for the  $\mathbf{e}_i$ -action coordinates  $(u_i^1, \dots, u_i^n)$ , by the formula of the coordinate transformation examined in the previous subsection, we have the same formula

$$(3.1) \quad (\text{Vol}(\mu^{-1}(p)))^2 = C \left( 1 - \sum_{j=1}^n u_i^j \right) \prod_{k=1}^n u_i^k.$$

**3.3. Proof of the main theorem.** Here we give a property of a moment polytope which holds only for  $\mathbb{C}P^n$  among compact toric Kähler manifolds.

**Lemma 10.** *Let  $\mathbf{u}_i = (u_i^1, \dots, u_i^n)$  be the  $\mathbf{e}_i$ -action coordinates of  $p \in \text{Int}(\Delta)$ . Then there exists a number  $i$  such that  $|||\mathbf{u}_i||| \leq 1$ .*

*Proof.* Suppose that  $|||\mathbf{u}_0||| > 1$ . By definition, there exists  $i \in \{0, 1, \dots, n\}$  such that  $\bar{\mathbf{u}}_0 = u_0^i$ . Then we have

$$\begin{aligned} \mathbf{u}_i &= (u_i^1, \dots, u_i^{i-1}, u_i^i, u_i^{i+1}, \dots, u_i^n) \\ &= (u_0^1, \dots, u_0^{i-1}, 1 - |\mathbf{u}_0|, u_0^{i+1}, \dots, u_0^n), \end{aligned}$$

and hence  $|\mathbf{u}_i| = 1 - u_0^i$ . Therefore,

$$|||\mathbf{u}_i||| = 1 - u_0^i + \bar{\mathbf{u}}_i = \begin{cases} 1 - u_0^i + u_0^j \leq 1 & (\text{if } \bar{\mathbf{u}}_i = u_0^j \ (i \neq j)) \\ 2 - |||\mathbf{u}_0||| < 1 & (\text{if } \bar{\mathbf{u}}_i = 1 - |\mathbf{u}_0|), \end{cases}$$

which implies  $|||\mathbf{u}_i||| \leq 1$ . □

Theorem 5 is a direct consequence of the following



**Theorem 11.** *Let  $(\mathbb{C}P^n, \omega_{\text{FS}})$  be the  $n$ -dimensional complex projective space. Let  $\mu : \mathbb{C}P^n \rightarrow \Delta$  be the moment map associated with the standard Hamiltonian  $T^n$ -action on  $\mathbb{C}P^n$ . Pick a point  $p \in \text{Int}(\Delta)$  and take  $\mathbf{e}_i$ -action coordinates  $\mathbf{u}_i = (u_i^1, \dots, u_i^n) \in \text{Int}(\Delta)$  of  $p$  which satisfies that  $\|\mathbf{u}_i\| \leq 1$ . If  $N(\mathbf{u}_i) \geq 3$ , then the Lagrangian torus orbit  $\mu^{-1}(p) \subset \mathbb{C}P^n$  is not Hamiltonian volume minimizing.*

**Remark 12.** *We denote by  $D_n$  the set of all points in  $\text{Int}(\Delta)$  which satisfy the assumption of Theorem 11. Of course,  $D_n$  is open dense in  $\text{Int}(\Delta)$  if  $n \geq 3$ .*

*Proof.* Firstly, since  $\|\mathbf{u}_i\| < \|\mathbf{u}_i\| \leq 1$ , by virtue of Theorem 9, we can take  $\mathbf{a} := \mathbf{u}_i$  in the proof of Proposition 8 and there exist a positive number  $\varepsilon > 0$  and a smooth function  $H : [0, 1] \times \mathbb{C}^n \rightarrow \mathbb{R}$  satisfying that

- (1)  $\text{Supp}(H) \subset [0, 1] \times \mu_0^{-1}(\tilde{\Delta}_{1-\varepsilon})$ ,
- (2)  $\phi_H^1(\mu_0^{-1}(\mathbf{u}_i)) = \mu_0^{-1}(\mathbf{u}'_i)$ .

Let  $p'$  be the element in  $\text{Int}(\Delta)$  whose  $\mathbf{e}_i$ -action coordinate is  $\mathbf{u}'_i$ . Notice that we have the identification  $(U_i, \omega_{\text{FS}}|_{U_i}, \mu|_{U_i}) \cong (\mu_0^{-1}(\tilde{\Delta}_1), \omega_0, \mu_0)$  as Hamiltonian  $T^n$ -spaces. Denoting it by  $\Phi : U_i \rightarrow \mu_0^{-1}(\tilde{\Delta}_1) \subset \mathbb{C}^n$ , we can define the following Hamiltonian function on  $\mathbb{C}P^n$ :

$$\hat{H}(t, x) := \begin{cases} H(t, \Phi(x)) & , \quad x \in U_i \\ 0 & , \quad x \in \mathbb{C}P^n \setminus U_i. \end{cases}$$

Then  $\hat{H} \in C_c^\infty([0, 1] \times \mathbb{C}P^n)$  and we obtain  $\phi_{\hat{H}}^1(\mu^{-1}(p)) = \mu^{-1}(p')$ .

Secondly, let us compare their volume. By assumption, there exist numbers  $a, b \in \{1, \dots, n\}$  such that  $\underline{\mathbf{u}}_i < u_i^a < u_i^b$ . Then by (3.1) we obtain

$$\begin{aligned} & (\text{Vol}(\mu^{-1}(p)))^2 - (\text{Vol}(\mu^{-1}(p')))^2 \\ &= C \left( 1 - \sum_{j=1}^n u_i^j \right) \prod_{k=1}^n u_i^k \\ &\quad - C \left( 1 - \sum_{j=1}^n u_i^j + u_i^b - (u_i^b - u_i^a + \underline{\mathbf{u}}_i) \right) \frac{u_i^b - u_i^a + \underline{\mathbf{u}}_i}{u_i^b} \prod_{k=1}^n u_i^k \\ &= C \frac{\prod_{k=1}^n u_i^k}{u_i^b} \left\{ u_i^b \left( 1 - \sum_{j=1}^n u_i^j \right) - (u_i^b - u_i^a + \underline{\mathbf{u}}_i) \left( 1 - \sum_{j=1}^n u_i^j + u_i^a - \underline{\mathbf{u}}_i \right) \right\} \\ &= C \frac{\prod_{k=1}^n u_i^k}{u_i^b} (u_i^a - \underline{\mathbf{u}}_i) \left( 1 - \sum_{j=1}^n u_i^j - u_i^b + u_i^a - \underline{\mathbf{u}}_i \right). \end{aligned}$$

Since  $\frac{\prod_{k=1}^n u_i^k}{u_i^b}(u_i^a - \underline{u}_i) > 0$  and

$$1 - \sum_{j=1}^n u_i^j - u_i^b + u_i^a - \underline{u}_i \geq 1 - |||\mathbf{u}_i||| + u_i^a - \underline{u}_i > 0$$

hold, we conclude that  $\text{Vol}(\mu^{-1}(p)) > \text{Vol}(\mu^{-1}(p'))$ .  $\square$

#### 4. THE CASE OF TORIC KÄHLER MANIFOLDS

In this section, we attempt to generalise the argument of Section 3 to toric Kähler manifolds. From now on, let  $(M, \omega, J)$  be a complex  $n$ -dimensional compact toric Kähler manifold, i.e.,  $M$  admits an effective holomorphic action of the complex torus  $(\mathbb{C}^\times)^n$  such that the restriction to the real torus  $T^n$  is Hamiltonian with respect to the Kähler form  $\omega$ . Its moment map is denoted by  $\mu : M \rightarrow \Delta = \mu(M) \subset \mathbb{R}^n$ . We may assume, without loss of generalities, that the moment polytope  $\Delta$  satisfies

$$\Delta = \{\mathbf{a} \in (\mathbb{R}_{\geq 0})^n \mid l_r(\mathbf{a}) := \langle \mathbf{a}, \mu_r \rangle - \lambda_r \geq 0, \lambda_r < 0, r = n+1, \dots, d\},$$

where each  $\mu_r$  is a primitive element of the lattice  $\mathbb{Z}^n \subset \mathbb{R}^n$  and inward-pointing normal to the  $r$ -th  $(n-1)$ -dimensional face of  $\Delta$ . It is known that each fibre  $\mu^{-1}(\mathbf{a})$ ,  $\mathbf{a} \in \text{Int}(\Delta)$ , is a Lagrangian torus and Hamiltonian minimal (see, e.g., [10, Proposition 3.1]).

The point  $\mu^{-1}(0) \in M$  is a fixed point of the  $(\mathbb{C}^\times)^n$ -action. By the construction, there exists a toric affine neighbourhood  $U$  of  $\mu^{-1}(0)$  such that  $(U, \mu^{-1}(0))$  is isomorphic to  $(\mathbb{C}^n, 0)$  as  $(\mathbb{C}^\times)^n$ -spaces. Using this identification we can define the standard complex coordinates  $(w^1, \dots, w^n)$  on  $U$ . Their polar coordinates are given by  $w^i = r^i e^{\sqrt{-1}\theta^i}$ ,  $i = 1, \dots, n$ .

As a set  $U$  is described as

$$U = M \setminus \mu^{-1}(\mathcal{F}), \quad \mathcal{F} := \bigcup_{F: \text{facet of } \Delta, 0 \notin F} F.$$

The restriction of the Kähler form  $\omega$  on  $U$  can be expressed as

$$\omega|_U = 2\sqrt{-1}\partial\bar{\partial}\varphi,$$

where  $\varphi$  is a real-valued function defined on  $(\mathbb{R}_{\geq 0})^n$  (see [1], [5]). Then the moment map  $\mu : M \rightarrow \Delta$  is represented as

$$\mu(p) = \left( r^1 \frac{\partial \varphi}{\partial r^1}, \dots, r^n \frac{\partial \varphi}{\partial r^n} \right) (p) =: (u^1, \dots, u^n)$$

Putting  $x^i := \sqrt{2u^i} \cos \theta^i$  and  $y^i := \sqrt{2u^i} \sin \theta^i$ , a straightforward calculation yields

$$\omega|_U = \sum_{i=1}^n dx^i \wedge dy^i = \sum_{i=1}^n du^i \wedge d\theta^i, \quad \mu|_U = \frac{1}{2} \sum_{i=1}^n \{(x^i)^2 + (y^i)^2\} \mathbf{e}_i$$

on  $U$ . Thus  $(U, \omega|_U, \mu|_U)$  is isomorphic as Hamiltonian  $T^n$ -spaces to  $(V, \omega_{0|V}, \mu_{0|V})$ , where  $V := \mu_0^{-1}(\Delta \setminus \mathcal{F})$  and  $\mu_0$  is the moment map defined in Section 1.

Now we are in a position to prove our second result (Theorem 6).

**Theorem 13.** *Let  $(M, \omega, J)$  be a complex  $n$ -dimensional compact toric Kähler manifold equipped with the moment map  $\mu : M \rightarrow \Delta \subset \mathbb{R}^n$  that is specified as above. Assume that  $n \geq 3$  and define a constant  $s_0 > 0$  as*

$$s_0 = \sup\{s > 0 \mid \tilde{\Delta}_s \subset \Delta\}.$$

*For  $\mathbf{a} \in \text{Int}(\Delta)$  with  $N(\mathbf{a}) \geq 3$ , if  $\|\mathbf{a}\| < s_0$ , then there exists  $\mathbf{a}' \in \text{Int}(\Delta)$  such that*

$$\phi(\mu^{-1}(\mathbf{a})) = \mu^{-1}(\mathbf{a}')$$

*for some  $\phi \in \text{Ham}(M, \omega)$ . Furthermore, if  $\|\mathbf{a}\|$  is sufficiently close to 0, then in addition these Lagrangian tori  $\mu^{-1}(\mathbf{a})$  and  $\mu^{-1}(\mathbf{a}')$  satisfy*

$$\text{Vol}(\mu^{-1}(\mathbf{a})) > \text{Vol}(\mu^{-1}(\mathbf{a}')).$$

*In particular, the above Lagrangian torus  $\mu^{-1}(\mathbf{a})$  is not Hamiltonian volume minimizing in  $M$ .*

*Proof.* Given a vector  $\mathbf{a} \in \text{Int}(\Delta)$  satisfying that  $N(\mathbf{a}) \geq 3$  and  $\|\mathbf{a}\| < s_0$ , according to Theorem 9, the proof of Proposition 8 enables us to take  $\mathbf{a}' \in \text{Int}(\Delta)$  and  $\{\phi_H^t\}_{0 \leq t \leq 1} \subset \text{Ham}_c(\mathbb{C}^n, \omega_0)$  which satisfy

$$(4.2) \quad \phi_H^1(\mu_0^{-1}(\mathbf{a})) = \mu_0^{-1}(\mathbf{a}'), \quad \text{Supp}(H) \subset [0, 1] \times \mu_0^{-1}(\tilde{\Delta}_{s_0})$$

and

$$(4.3) \quad \prod_{i=1}^n a_i > \prod_{i=1}^n a'_i.$$

Using the action angle coordinates  $(u^1, \dots, u^n, \theta^1, \dots, \theta^n)$  on  $U$  explained before, we identify  $(U, \omega|_U, \mu|_U)$  with  $(V, \omega_{0|V}, \mu_{0|V})$  and extend the Hamiltonian function  $H$  on  $U$  to  $M$  as

$$\hat{H}(t, x) := \begin{cases} H(t, x) & , \quad x \in U \\ 0 & , \quad x \in M \setminus U. \end{cases}$$

Then  $\hat{H} \in C^\infty([0, 1] \times M)$  and hence we obtain  $\phi_{\hat{H}}^1(\mu^{-1}(\mathbf{a})) = \mu^{-1}(\mathbf{a}')$ .

In order to complete the proof of Theorem 6, we have to compare the volume of two flat tori  $\mu^{-1}(\mathbf{a})$  and  $\mu^{-1}(\mathbf{a}')$  with respect to the induced metric from the toric Kähler manifold  $(M, \omega, J)$ .

In general, all  $\omega$ -compatible toric complex structures on  $(M, \omega)$  can be parametrized by smooth functions on  $\text{Int}(\Delta)$ , which is shown by Abreu in [1, Section 2]. More precisely, we can choose a strictly convex function  $g \in C^\infty(\text{Int}(\Delta))$  whose Hessian  $\text{Hess}_x(g)$  describes the complex structure  $J$  on  $M$ , and the determinant of  $\text{Hess}_x(g)$  is given by

$$\left\{ \delta(x) \prod_{r=1}^d l_r(x) \right\}^{-1},$$

where  $\delta \in C^\infty(\Delta)$  is a strictly positive function (see [1, Theorem 2.8]). Then the Riemannian metric of the fibre  $\mu^{-1}(p) \subset M$  of  $p \in \text{int}(\Delta)$  is given by the  $(n \times n)$ -matrix  $(\text{Hess}_x(g))^{-1}$ , and hence

$$\text{Vol}(\mu^{-1}(\mathbf{a}))^2 = (2\pi)^{2n} \delta(\mathbf{a}) \prod_{i=1}^n a_i \prod_{r=n+1}^d l_r(\mathbf{a})$$

holds. However, in general it is difficult to compare  $\text{Vol}(\mu^{-1}(\mathbf{a}))$  with  $\text{Vol}(\mu^{-1}(\mathbf{a}'))$  from this expression. So we introduce a parameter  $c \in (0, 1]$  and consider the volume of a Lagrangian torus  $\mu^{-1}(c\mathbf{a})$ . Then we obtain

$$\begin{aligned} & \text{Vol}(\mu^{-1}(c\mathbf{a}))^2 - \text{Vol}(\mu^{-1}(c\mathbf{a}'))^2 \\ &= (2\pi\sqrt{c})^{2n} \left\{ \delta(c\mathbf{a}) \prod_{i=1}^n a_i \prod_{r=n+1}^d l_r(c\mathbf{a}) - \delta(c\mathbf{a}') \prod_{i=1}^n a'_i \prod_{r=n+1}^d l_r(c\mathbf{a}') \right\}. \end{aligned}$$

The value at  $c = 0$  of the quantity of the inside of the brackets is

$$\delta(\mathbf{0}) \prod_{r=n+1}^d (-\lambda_r) \left( \prod_{i=1}^n a_i - \prod_{i=1}^n a'_i \right),$$

which is positive due to (4.3). Therefore, there exists a constant  $c_{\mathbf{a}} > 0$  such that

$$\text{Vol}(\mu^{-1}(c\mathbf{a})) - \text{Vol}(\mu^{-1}(c\mathbf{a}')) > 0.$$

holds for any  $c \in (0, c_{\mathbf{a}})$ . Thus we complete the proof.  $\square$

## 5. REMAINED OPEN PROBLEMS

Finally, let us discuss the remained part of Oh's conjecture and add some remarks. According to Corollary 3, the unsolved part of Conjecture 2 is as follows.

**Problem 14.** Let  $0 < a \leq b$  and  $k = 1, 2, \dots, n$ . Is a product torus  $T(\underbrace{a, \dots, a}_k, \underbrace{b, \dots, b}_{n-k})$  in  $\mathbb{C}^n$  Hamiltonian volume minimizing?

This problem had already been considered by Anciaux in the case where  $n = 2$ , and he gave a partial answer to it. He showed in [2, Main Theorem] that  $T(a, a) \subset \mathbb{C}^2$  has the least volume among all *Hamiltonian minimal* Lagrangian tori of its Hamiltonian isotopy class. However, this result does not imply that  $T(a, a)$  is Hamiltonian volume minimizing in  $\mathbb{C}^2$ .

Next we turn to the case of  $\mathbb{C}P^n$ . We proved in Theorem 5 that every Lagrangian torus that is the preimage of a point in  $D_n \subset \text{Int}(\Delta)$  is not Hamiltonian volume minimizing. However, the barycentre  $p_0$  of  $\Delta$  is *not* in  $D_n$ . The corresponding fibre  $\mu^{-1}(p_0) \subset \mathbb{C}P^n$  is a minimal Lagrangian torus and called the *Clifford torus*. Thus the following question raised by Oh is still open.

**Problem 15** ([8], p. 516). Is the Clifford torus in  $\mathbb{C}P^n$  Hamiltonian volume minimizing?

We point out that Urbano proved that the only Hamiltonian stable minimal Lagrangian torus in  $\mathbb{C}P^2$  is the Clifford one (see [11, Corollary 2]).

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